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$U^\mu = \frac{dx^\mu}{d\tau}$ weirdness: $U^\mu U_\mu = \eta_{\mu\nu} U^\mu U^\nu$

$$= \eta_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}$$
$$= \frac{\eta_{\mu\nu} dx^\mu dx^\nu}{d\tau^2}$$
$$= \frac{\eta_{\mu\nu} dx^\mu dx^\nu}{-\eta_{\mu\nu} dx^\mu dx^\nu}$$
$$= -1$$

WTF?!
e h u
l a n
l t n
s y

This might seem strange, usually $V_i V^i = V^2$ for $x^i(t)$.

But consider: "$\vec{V}$" $= \left\langle \frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds} \right\rangle$ ds = path length

"$\vec{V}\cdot\vec{V}$" $= \frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = \frac{ds^2}{ds^2} = 1$

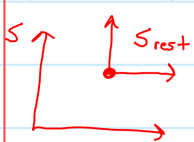
So this is 'just a consequence of how we are parameterizing paths.

Components of U^μ: $U^0(\tau) = \frac{dx^0}{d\tau} = \underset{c=1}{\frac{dt}{d\tau}} = \frac{\gamma\, dt_{rest}}{d\tau} = \gamma \frac{d\tau}{d\tau} = \gamma$

in S w/ (t,x,y,z) $U^i(\tau) = \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = v^i \gamma$

$\gamma \equiv \frac{1}{\sqrt{1-v^2}}$

velocity connecting S to S_{rest}



$$\Downarrow$$
$$\boxed{U^\mu = \begin{pmatrix}\gamma \\ \gamma\vec{v}\end{pmatrix}}$$

$\hookrightarrow U_\mu U^\mu = -\gamma^2 + \gamma^2 v^2 = -\frac{1}{1-v^2} + \frac{v^2}{1-v^2} = -1$

In S_{rest}: $U^\mu = \begin{pmatrix}1 \\ 0\end{pmatrix}$ (again $U_\mu U^\mu = -1$)

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For momentum: $p^i \to p^\mu \equiv \underbrace{m}_{\text{mass}} U^\mu = \begin{pmatrix} m\gamma \\ m\gamma\vec{v} \end{pmatrix}$

If $v^2 \ll \underbrace{1}_{c^2} \Rightarrow \gamma \simeq 1 + \frac{1}{2}v^2 + \cdots \Rightarrow \begin{cases} m\gamma \simeq \overset{\text{rest energy}}{m} + \overbrace{\frac{1}{2}mv^2}^{\text{non-relativistic KE}} + \cdots = E \\ m\gamma v^i \simeq \underbrace{mv^i}_{\text{non-relativistic momentum}} + \mathcal{O}(v^{i\,3}) + \cdots \equiv p^i \end{cases}$

relativistic energy and momentum!

So: $P^\mu = mU^\mu = \begin{pmatrix} E \\ \vec{p} \end{pmatrix}$

Then: $p_\mu p^\mu = m^2 U_\mu U^\mu = \underbrace{-m^2 = -E^2 + p^2}_{E^2 = p^2 + m^2}$ (w/ c $E^2 = p^2c^2 + m^2c^4$)

Even though our parameterization doesn't work when $m^2 = 0$, this result does!

$$p_\mu p^\mu = \begin{cases} <0 & m^2 > 0 & ds^2 < 0 & \text{timelike} \\ =0 & m^2 = 0 & ds^2 = 0 & \text{lightlike} \\ >0 & m^2 < 0 & ds^2 > 0 & \text{spacelike (tachyonic)} \end{cases}$$

We could try to relativize F^i to get $\Sigma F^\mu = \frac{dp^\mu}{d\tau}$ which will combine the work-energy and impulse-momentum theorems. But our primary concern is the gravitational force which will play out a bit differently.

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The next topic is going to seem like a strange focus at first, but remember that interesting sources of gravity are usually large, so instead of thinking one particle at a time, we should consider a large number of them.
The source of curvature/gravitation will be energy (including mass) and momentum.
We already have a quantity describing this, i.e. P^μ, but this is really only useful for one particle. We will now see how considering many particles leads us instead to the energy-momentum tensor $T^{\mu\nu}$.

Densities

let's work in an infinitesimal region, so everything is differentially small!

Like last lecture, we will get a lot of mileage out of insisting that quantities be tensors!

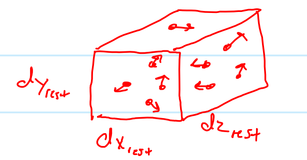
Example: Particle Number Density $dn = \frac{dN}{dV}$ ← scalar (dN)

dV ← $dx\,dy\,dz$

We can naturally define:

$$dn_{rest} \equiv \frac{dN}{dV_{rest}} = \#\text{ density when volume in question is at rest.}$$

dV_{rest} ← $dx_{rest}\,dy_{rest}\,dz_{rest}$



We can now ask if dn as defined is a tensor.

volume contracts but # is same

Consider boosting along x by $-v$: $dn \to dn' = \frac{dN}{\frac{1}{\gamma}dx\,dy\,dz} = \gamma\frac{dN}{dV} = \underbrace{\gamma dn > dn}$

$dn' = \gamma dn_{rest}$ if $dn = dn_{rest}$

dn is clearly not a scalar (it changed!) but neither is a vector, etc. It is not a tensor!
What is it? Well $dn' = \gamma dn_{rest}$ is similar to $dt' = \gamma dt_{rest}$ but dt is the 0-component of a 4-vector. So perhaps dn is the 0-component of a 4-vector as well.

$$dN^\mu \equiv dn_{rest}\, U^\mu = \begin{pmatrix} dn_{rest}\gamma \\ dn_{rest}\gamma v^i \end{pmatrix} = \begin{pmatrix} dn \\ v^i dn \end{pmatrix} \Rightarrow dN^\mu_{rest} = \begin{pmatrix} dn_{rest} \\ \vec{0} \end{pmatrix}$$

$dn_{rest} \equiv \frac{dN}{dV_{rest}}$ = constant

The moral to note: We started w/ a scalar dN, but to make a tensor density, we needed to introduce the vector number density dN^μ.

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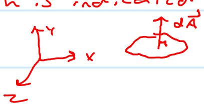
Is there a deeper explanation for why creating a tensor density seems to "raise" the tensor nature of what we start w/, i.e. dN-scalar $\ni dN^\mu$-vector.

For densities we need 3D volumes, but a 3D volume in 4D is labelled by its size dV and direction (think about 2D area in 3D) which is indicated by a normal dual vector $\Rightarrow$ $n_\mu dV$



n_μ: normal to 3D "surface"

The quick and dirty: In 3D a "surface" is defined by $f(x,y,z) = 0$

e.g. $x^2+y^2+z^2 = R^2 \Rightarrow f(x,y,z) = x^2+y^2+z^2-R^2 = 0$

Then $d\vec{A} = \underbrace{\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}}_{\text{defined w/ } \frac{\partial}{\partial x^i} = \partial_i \text{ so dual vector components}} = 2x\hat{i} + 2y\hat{j} + 2z\hat{k} = r\hat{r}$

You are familiar w/ $dN = \underbrace{\frac{dN}{dV}}_{\#\text{ density}} \overbrace{dV}^{\text{volume}}$, which now becomes $dN = \underbrace{dN^\mu}_{\#\text{ density}} \overbrace{n_\mu dV}^{\text{volume}} = $ scalar.

If we try to extend this to $\frac{dP^\mu}{dV}$ we find $dP^\mu = \underbrace{?}_{\text{momentum-density}} \overbrace{n_\nu dV}^{\text{volume}} \Rightarrow ? = T^{\mu\nu}$